University of California, Berkeley
Physics 105 Fall 2000 Section 1 (*Strovink*)

SOLUTION TO EXAMINATION 1

Directions. Do all problems (weights are indicated). This is a closed-book closed-note exam except for one $8\frac{1}{2} \times 11$ inch sheet containing any information you wish on both sides. You are free to approach the proctor to ask questions – but he will not give hints and will be obliged to write your question and its answer on the board. Roots, circular functions, *etc.*, may be left unevaluated if you do not know them. Use a bluebook. Do not use scratch paper – otherwise you risk losing part credit. Cross out rather than erase any work that you wish the grader to ignore. Justify what you do. Box or circle your answer.

1. (20 points)
In one dimension, the Lagrangian for a relativistic free electron is

$$\mathcal{L} = -mc^2\sqrt{1-\frac{\dot{x}^2}{c^2}}\ ,$$

where m is the electron mass and c is the speed of light.

Find the total energy of the electron in terms of m, c, and $\dot{x}$. Prove your result given only this Lagrangian, using no other knowledge of relativity.
Solution:

$$\begin{aligned}
\mathcal{H} &\equiv \dot{x}\frac{\partial\mathcal{L}}{\partial\dot{x}} - \mathcal{L} \\
&= \dot{x}\frac{-\frac{1}{2}mc^2}{\sqrt{1-\frac{\dot{x}^2}{c^2}}}\left(\frac{-2\dot{x}}{c^2}\right) + mc^2\sqrt{1-\frac{\dot{x}^2}{c^2}} \\
&= \frac{mc^2}{\sqrt{1-\frac{\dot{x}^2}{c^2}}}\left(\frac{\dot{x}^2}{c^2} + 1 - \frac{\dot{x}^2}{c^2}\right) \\
&= \frac{mc^2}{\sqrt{1-\frac{\dot{x}^2}{c^2}}} \\
\frac{d\mathcal{H}}{dt} &= -\frac{\partial\mathcal{L}}{\partial t} = 0\ ,
\end{aligned}$$

so the Hamiltonian $\mathcal{H}$ is constant and equal to E, the total energy. Thus

$$E = \frac{mc^2}{\sqrt{1-\frac{\dot{x}^2}{c^2}}}\ .$$

2. (25 points)
During $-\infty < t < 0$, a linear oscillator satisfying the equation of motion

$$\ddot{x} + \omega_0\dot{x} + \omega_0^2 x = \frac{F_x(t)}{m}$$

is driven at its resonant frequency by a force per unit mass

$$\frac{F_x(t)}{m} = a_0\cos\omega_0 t\ ,$$

where a_0 is a constant.
(**a**) (10 points)
Find $x(0)$ and $\dot{x}(0)$ at $t = 0$.
Solution:
Since the driving force was first applied long ago at $t = -\infty$, the effects of that initial transient have died out and can be ignored; for $t < 0$ all we need is a particular solution. To get it, as usual we substitute

$$x = \mathrm{Re}\big(A\exp{(i\omega_0 t)}\big)$$

into the differential equation and choose to solve the complex version of the result, rather than its real part. Cancelling the common factor $\exp{(i\omega_0 t)}$, we obtain

$$\begin{aligned}
(-\omega_0^2 + i\omega_0^2 + \omega_0^2)A &= a_0 \\
A &= -\frac{ia_0}{\omega_0^2} \\
x(t<0) &= \frac{a_0}{\omega_0^2}\sin\omega_0 t \\
x(0) &= 0 \\
\dot{x}(0) &= \frac{a_0}{\omega_0}\ .
\end{aligned}$$

(**b**) (15 points)
At $t = 0$ the driving force is turned off. Find $x(t)$ for $t > 0$.
Solution:
Here we need a solution $x_h(t)$ to the homogeneous equation. Substituting

$$x_h = \mathrm{Re}\big(B \exp(i\omega t)\big)$$

with ω a constant to be determined, and cancelling the common factor $\exp(i\omega t)$, we obtain

$$\begin{aligned}
0 &= -\omega^2 + i\omega_0\omega + \omega_0^2 \\
\omega &= \frac{i\omega_0 \pm \sqrt{-\omega_0^2 + 4\omega_0^2}}{2} \\
&= -\frac{i\omega_0}{2} \pm \sqrt{\tfrac{3}{4}}\omega_0 \\
x_h(t) &= B \exp(-\tfrac{1}{2}\omega_0 t) \cos\left(\sqrt{\tfrac{3}{4}}\omega_0 t + \beta\right) ,
\end{aligned}$$

where B and β are adjustable constants. (This standard underdamped solution may also simply be recalled from memory or from notes.) Enforcing the initial condition $x(0) = 0$, we take $\beta = \frac{\pi}{2}$ and the solution becomes

$$x(t) = -B \exp(-\tfrac{1}{2}\omega_0 t) \sin\left(\sqrt{\tfrac{3}{4}}\omega_0 t\right) .$$

Matching the remaining initial condition,

$$\begin{aligned}
\frac{a_0}{\omega_0} &= \dot{x}(0) \\
&= B\sqrt{\tfrac{3}{4}}\omega_0 \\
\sqrt{\tfrac{4}{3}}\frac{a_0}{\omega_0^2} &= B \\
x(t > 0) &= -\sqrt{\tfrac{4}{3}}\frac{a_0}{\omega_0^2} \exp(-\tfrac{1}{2}\omega_0 t) \sin\left(\sqrt{\tfrac{3}{4}}\omega_0 t\right) .
\end{aligned}$$

3. (35 points)
A small bead of mass m is constrained to move without friction on a circular hoop of radius a that rotates with constant angular velocity Ω about a vertical diameter. Use θ, the polar angle of the bead, as the single generalized coordinate ($\theta = 0$ at the bottom). Do not neglect gravity.
(**a**) (5 points)
Write the Lagrangian as a function of θ and $\dot{\theta}$. Remember to take into account the two different components of the bead's velocity.
Solution:
The bead's velocity along the hoop ($a\dot{\theta}$) is orthogonal to the velocity associated with the hoop's rotation ($a \sin\theta\Omega$). From the center of the hoop, the height of the bead is $-a\cos\theta$. So the Lagrangian is

$$\begin{aligned}
\mathcal{L} &= T - U \\
&= \tfrac{1}{2}ma^2(\dot{\theta}^2 + \Omega^2 \sin^2\theta) + mga\cos\theta .
\end{aligned}$$

(**b**) (5 points)
Obtain the differential equation of motion for θ.
Solution:

$$\begin{aligned}
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} &= \frac{\partial \mathcal{L}}{\partial \theta} \\
\frac{d}{dt}(ma^2\dot{\theta}) &= ma^2\Omega^2 \sin\theta\cos\theta - mga\sin\theta \\
0 &= \ddot{\theta} - \Omega^2 \sin\theta\cos\theta + \frac{g}{a}\sin\theta .
\end{aligned}$$

(**c**) (10 points)
Find a restriction on Ω such that small oscillations about $\theta = 0$ can occur. What is the angular frequency of these oscillations?
Solution:
To first order in $\theta \ll 1$, $\sin\theta \approx \theta$ and $\cos\theta \approx 1$. The equation of motion becomes

$$0 = \ddot{\theta} + \big(\frac{g}{a} - \Omega^2\big)\theta ,$$

which describes simple harmonic motion of θ with angular frequency

$$\omega_0 = \sqrt{\frac{g}{a} - \Omega^2} ,$$

provided that

$$\Omega < \sqrt{\frac{g}{a}} .$$

(**d**) (5 points)
If Ω does not obey the restriction in part (**c**), about what other equilibrium position(s) can the bead undergo small oscillations?

Solution:
At equilibrium, $\ddot{\theta} = 0$, so the equation of motion yields

$$0 = -\Omega^2 \sin\theta \cos\theta + \frac{g}{a} \sin\theta \, .$$

From the results of part (**c**), this equilibrium point occurs away from $\theta = 0$, so we can cancel the common factor $\sin\theta$. Then the equilibrium coordinate θ_0 becomes

$$\begin{aligned} \cos\theta_0 &= \frac{g}{a\Omega^2} \\ \theta_0 &= \pm\left|\arccos\left(\frac{g}{a\Omega^2}\right)\right| \, . \end{aligned}$$

(**e**) (10 points)
What is the angular frequency of the small oscillations to which part (**d**) refers?
Solution:
Applying the method of perturbations, substituting $\theta = \theta_0 + \psi$, we recalculate the angular factors to first order in ψ:

$$\begin{aligned} \theta &= \theta_0 + \psi \\ \sin\theta &\approx \sin\theta_0 + \psi\cos\theta_0 \\ \cos\theta &\approx \cos\theta_0 - \psi\sin\theta_0 \\ \sin\theta\cos\theta &\approx \sin\theta_0\cos\theta_0 + \psi(\cos^2\theta_0 - \sin^2\theta_0) \, . \end{aligned}$$

Applying these substitutions to the equation of motion,

$$\begin{aligned} 0 &= \ddot{\theta} - \Omega^2 \sin\theta\cos\theta + \frac{g}{a}\sin\theta \\ &\approx \ddot{\psi} - \Omega^2\big(\sin\theta_0\cos\theta_0 + \psi(\cos^2\theta_0 - \sin^2\theta_0)\big) \\ &+ \frac{g}{a}\big(\sin\theta_0 + \psi\cos\theta_0\big) \, . \end{aligned}$$

As usual, substituting $\cos\theta_0 = g/a\Omega^2$ from part (**d**) allows the terms independent of ψ to cancel:

$$0 = \ddot{\psi} - \Omega^2\psi(\cos^2\theta_0 - \sin^2\theta_0) + \frac{g}{a}\psi\cos\theta_0 \, .$$

The same substitution allows the terms containing $\cos\theta_0$ to cancel:

$$0 = \ddot{\psi} + \Omega^2\psi\sin^2\theta_0 \, .$$

This is an equation of simple harmonic motion for ψ with angular frequency

$$\begin{aligned} \omega_{\text{osc}} &= \Omega\sin\theta_0 \\ &= \Omega\sqrt{1-\cos^2\theta_0} \\ &= \Omega\sqrt{1-\frac{g^2}{a^2\Omega^4}} \, . \end{aligned}$$

4. (20 points)
An antiproton with mass m and charge $-e$ is incident upon a nucleus with mass M and charge Ze. You may assume them to be point particles. The nucleus is initially at rest. When m is still very far from M, it has velocity v_0, directed so that the two masses would miss by a distance b if there were no attraction between them. *State the system of units in which you are working (SI or cgs).*
(**a**) (10 points)
Obtain a pair of equations which, if solved, would allow you to calculate the distance of closest approach between the antiproton and the nucleus.
Solution:
We shall work in SI. Initially the angular momentum of the two particles about their common center of mass is $\mu v_0 b$, where $\mu = mM/(m+M)$ is the reduced mass. At the perigee, when the particles are separated by a distance $r_{\min}$ and their relative velocity has magnitude $v_{\max}$, the angular momentum is the same because the force is central. Therefore

$$\begin{aligned} \mu v_{\max} r_{\min} &= \mu v_0 b \\ r_{\min} &= b\frac{v_0}{v_{\max}} \, . \end{aligned}$$

Initially, because the particles are greatly separated, their potential energy is zero. Therefore the initial energy has only a kinetic term, $\frac{1}{2}\mu v_0^2$. At the perigee, by energy conservation,

$$\tfrac{1}{2}\mu v_0^2 = \tfrac{1}{2}\mu v_{\max}^2 - \frac{Ze^2}{4\pi\epsilon_0 r_{\min}} \, .$$

This is a pair of equations that may be solved for the two unknowns $r_{\min}$ and $v_{\max}$.
(**b**) (10 points)
As v_0 approaches zero, through what angle will

the antiproton scatter? (Elementary reasoning, if stated correctly, should be sufficient here.)

Solution:

As $v_0 \to 0$, the total energy approaches zero as well. An orbit of zero total energy is a parabola, which has parallel asymptotes. Therefore the scattering angle will approach $180°$.